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On equitable coloring of corona of wheels

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Abstract

The notion of equitable colorability was introduced by Meyer in 1973 [9]. In this paper we obtain interesting results regarding the equitable chromatic number $\chi_=$ for the corona graph of a simple graph with a wheel graph $G \circ W_n$. Some extensions into l-corona products are also determined.

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1. Introduction

If the set of vertices of a graph G can be partitioned into k classes $V_1, V_2, \dots, V_k$ such that each V_i is an independent set and the condition $||V_i| - |V_j|| \leq 1$ holds for every pair (i, j), then G is said to be *equitably k-colorable*. The smallest integer k for which G is equitably k-colorable is known as the *equitable chromatic number* [9] of G and denoted by $\chi_=(G)$. This subject is widely discussed in literature [1, 4, 6, 7, 9]. In general, the problem of optimal equitable coloring, in the sense of number color used, is NP-hard.

The *corona* of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. For any integer $l \geq 2$, we define the graph $G_1 \circ^l G_2$ recursively from $G_1 \circ G_2$ as

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$G_1 \circ^l G_2 = (G_1 \circ^{l-1} G_2) \circ G_2$. Graph $G_1 \circ^l G_2$ is also named as l-corona product of G_1 and G_2. This kind of product was introduced by Harary and Frucht in 1970 [2].

Even more, we know [4] that the problem of the equitable coloring of corona graphs $G \circ H$ is NP-hard when G is 4-regular graph and $H = K_2$. So we have to look for simplified structure of graphs allowing polynomial-time algorithms. This paper gives such solutions for corona graph of a simple graph with a wheel graph. Some extensions for l-corona products are also determined. This way we confirm Equitable Coloring Conjecture posed by Meyer [9] for graphs under consideration.

Conjecture 1 (Equitable Coloring Conjecture (ECC) [9]). *For any connected graph G, other than the complete graph or odd cycle, $\chi_=(G) \leq \Delta(G)$.*

This conjecture has been verified for all graphs with six or fewer vertices. Lih and Wu [7] proved that the Equitable Coloring Conjecture (ECC) is true for all bipartite graphs. Wang and Zhang [10] considered a broader class of graphs, namely r-partite graphs. They proved that Meyer's conjecture is true for complete graphs from this class. The conjecture (or even the stronger one) was confirmed for outerplanar graphs [11] and planar graphs with maximum degree at least 13 [12].

Graph products are interesting and useful in many situations [5]. Equitable coloring of Cartesian, weak tensor and strong tensor products for some classes of graphs was considered in [3, 8].

For any integer $n \geq 4$, the *wheel graph* W_n is the $n-$vertex graph obtained by joining a vertex v_1 to each of the $n-1$ vertices $\{w_1, w_2, \ldots w_{n-1}\}$ of the cycle graph C_{n-1}.

2. Equitable coloring on corona graph of simple graph with wheel graph

We start with giving results for coronas of a single vertex and a wheel.

Theorem 2.1. *Let n be a positive integer, $n \geq 4$. Then*

$$\chi_=(K_1 \circ W_n) = \left\lceil \frac{n-1}{2} \right\rceil + 2.$$

Proof. The color used for coloring the vertex of K_1 and the color used for coloring vertex v_1 cannot be used more times, so we can use any other color at most twice. Hence the value of the equitable chromatic number is equal to $\left\lceil \frac{n-1}{2} \right\rceil + 2$. □

We notice that $\Delta(K_1 \circ W_n) = n \geq \lceil (n-1)/2 \rceil + 2$ for $n \geq 4$. This means that ECC holds for $K_1 \circ W_n$, $n \geq 4$.

Next, we consider coronas, where the set of vertices of graph G includes more than one element.

Theorem 2.2. *Let G be an equitably 4-colorable graph on, $m \geq 2$, vertices and let m be even, n be odd, and $n \geq 4$, then*

$$\chi_=(G \circ W_n) = 4.$$

Proof. Let $n_i(k)$ be the number of appearance of color k, $1 \leq k \leq 4$, in the ith copy of W_n corresponding to vertex u_i of G in $G \circ W_n$, $i = 1, 2, \ldots, m$.

Let $f(u_i) = j$ be the color assigned to vertex u_i $(1 \leq i \leq m)$ of G. Since G is 4-colorable j takes the values in the range $1 \leq j \leq 4$.

We color graph G equitably with four colors. We order the vertices of G: $u_1, u_2, \ldots, u_m$ in such a way that vertex u_i is colored with color $i \bmod 4$ - we use color 4 instead of color 0 (in some cases recoloring is needed). We extend this coloring into whole graph $G \circ W_n$ due to the following conditions. We consider two cases:

1. $m \bmod 4 \equiv 0$
   If $f(u_i) = j$, $u_i \in V(G)$, $1 \leq j \leq 4$, then
   - $n_i((j+1) \bmod 4) = 1$,
   - $n_i((j+2) \bmod 4) = \dfrac{n-1}{2}$ and
   - $n_i((j+3) \bmod 4) = \dfrac{n-1}{2}$.

   In the above coloring, we use each color exactly $(n+1)m/4$ times. Graph $G \circ W_n$ is colored equitably.
2. $m \bmod 4 \equiv 2$
   We color first $m-2$ copies of W_n as we have colored the corresponding vertices in Case (1). We color last two copies in the following way. For each vertex $u_i, i = m-1, m$, if $f(u_i) = j$, $1 \leq j \leq 2$, then the extended coloring must fulfill the following conditions.
   - $n_i\left((j+2)\right) = 1$,
   - $n_i\left((j+3) \bmod 4\right) = \dfrac{n-1}{2}$,
   - $n_i\left(j+1\right) = \dfrac{n-1}{2}$.

   We use each of four colors exactly $(n+1)\lfloor m/4 \rfloor + (n+1)/2$ times. Graph $G \circ W_n$ is colored equitably.

Hence $\chi_=(G \circ W_n) \leq 4$. By the definition of corona graph, graph $G \circ W_n$ contains K_4. Hence $\chi_=(G \circ W_n) = 4$. □

Theorem 2.3. *Let G be an equitably 4-colorable graph on 5 vertices, then*

$$\chi_=(G \circ W_5) = 4.$$

Proof. Since W_5 has the cycle C_4, $\chi(W_4) \geq 3$. By the definition of corona, each vertex u_i of G is adjacent to every vertex of its copy of W_n. Hence $\chi_=(G \circ W_5) \geq 4$.

By assigning the colors 1,2,3 and 4 as given below, it is concluded that the 1 appears 7 times, 2 appears 8 times, 3 appears 8 times and 4 appears 7 times. (i.e) The difference between the number of appearance of each pair of colors does not exceed one. Hence $\chi_=(G \circ W_5) \leq 4$. Hence $\chi_=(G \circ W_5) = 4$. □

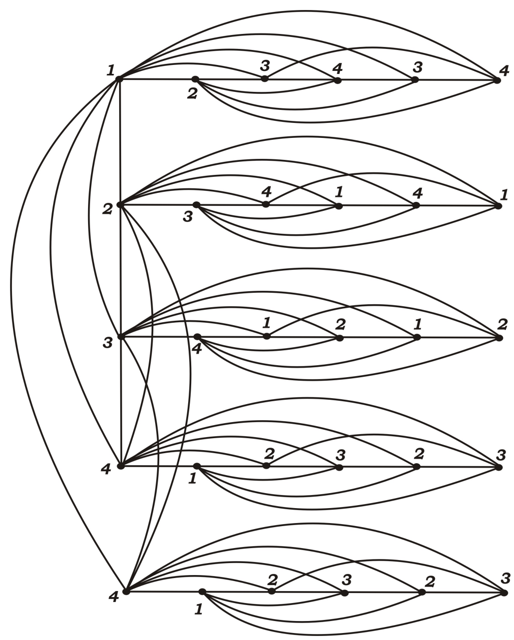

Figure 1. An equitable 4-coloring of $K_{1,1,1,2} \circ W_5$ with $n(1) = n(4) = 7$ and $n(2) = n(3) = 8$.

Now, we consider the remaining cases of m and n. It turns out that in these cases five colors are desirable for proper equitable coloring.

Theorem 2.4. *Let G be an equitably 5-colorable graph on m vertices. If $m \bmod 2 \equiv 1, n \geq 7$ or m, n even with $n \geq 4$ then*

$$\chi_=(G \circ W_n) = 5.$$

Proof. Let $n_i(k)$ be the number of appearance of color k, $1 \leq k \leq 5$, in the ith copy of W_n corresponding to vertex u_i of G in $G \circ W_n$, $i = 1, 2, \ldots, m$.

Let $f(u_i) = j$ be the color assigned to vertex u_i $(1 \leq i \leq m)$ of G. Since G is 5-colorable j takes the values in the range $1 \leq j \leq 5$.

We color graph G equitably with five colors. We order the vertices of G: $u_1, u_2, \ldots, u_m$ in such a way that vertex u_i is colored with color $i \bmod 5$ - we use color 5 instead of color 0 (in some cases recoloring is needed). We extend this coloring to the whole graph $G \circ W_n$ due to the following conditions. We consider five cases dependently on the value of m.

1. $m \bmod 5 \equiv 0$
   For each vertex $u_i \in V(G)$ if $f(u_i) = j$, $1 \leq j \leq 5$, then
   - $n_i((j+1) \bmod 5) = 1$,
   - $n_i((j+2) \bmod 5) = 1$,
   - $n_i((j+3) \bmod 5) = \left\lceil \dfrac{n-2}{2} \right\rceil$,

- $n_i((j+4) \bmod 5) = \left\lfloor \dfrac{n-2}{2} \right\rfloor$

We use each of the five colors exactly $(n+1)m/5$ times. Graph $G \circ W_n$ is colored equitably.

2. $m \bmod 5 \equiv 1$

First, we color $m-6$ copies of W_n as we color the corresponding vertices in Case (1). We color last six copies in the following way. For each vertex u_i $(m-5 \leq i \leq m)$ we extend the coloring due to the following conditions, dependently on n.

(a) $n \bmod 5 \equiv 0$

- For vertex u_{m-5} $(f(u_{m-5}) = 1)$ we have $n_{m-5}(2) = 1$, $n_{m-5}(3) = n_{m-5}(4) = \dfrac{2n-5}{5}$, $n_{m-5}(5) = \dfrac{n+5}{5}$.
- For vertex u_{m-4} $(f(u_{m-4}) = 2)$ we have $n_{m-4}(3) = 1$, $n_{m-4}(1) = n_{m-4}(5) = \dfrac{2n-5}{5}$, $n_{m-4}(4) = \dfrac{n+5}{5}$.
- For vertex u_{m-3} $(f(u_{m-3}) = 3)$ we have $n_{m-3}(4) = 1$, $n_{m-3}(2) = \dfrac{2n}{5}$, $n_{m-3}(5) = \dfrac{2n-10}{5}$, $n_{m-3}(1) = \dfrac{n+5}{5}$.
- For vertex u_{m-2} $(f(u_{m-2}) = 4)$ we have $n_{m-2}(5) = 1$, $n_{m-2}(3) = \dfrac{2n}{5}$, $n_{m-2}(2) = \dfrac{2n-5}{5}$, $n_{m-2}(1) = \dfrac{n}{5}$.
- For vertex u_{m-1} $(f(u_{m-1}) = 5)$ we have $n_{m-1}(3) = 1$, $n_{m-1}(1) = n_{m-1}(2) = \dfrac{2n-5}{5}$, $n_{m-1}(4) = \dfrac{n+5}{5}$.
- For vertex u_m $(f(u_m) = 1)$ we have $n_m(2) = 1$, $n_m(3) = n_m(4) = \dfrac{2n-5}{5}$, $n_m(5) = \dfrac{n+5}{5}$

Each of the colors 1, 2, 3 and 5 are used $(6n+5)/5$ times and color 4 is used $(6n+5)/5+1$ times.

(b) $n \bmod 5 \equiv 1$ or $n \bmod 5 \equiv 4$

- For vertex u_{m-5} $(f(u_{m-5}) = 1)$ we have $n_{m-5}(2) = 1$, $n_{m-5}(3) = n_{m-5}(4) = \left\lfloor \dfrac{2n}{5} \right\rfloor$, $n_{m-5}(5) = \left\lceil \dfrac{n-1}{5} \right\rceil$.
- For vertex u_{m-4} $(f(u_{m-4}) = 2)$ we have $n_{m-4}(3) = 1$, $n_{m-4}(1) = n_{m-4}(5) = \left\lfloor \dfrac{2n}{5} \right\rfloor$, $n_{m-4}(4) = \left\lceil \dfrac{n-1}{5} \right\rceil$.
- For vertex u_{m-3} $(f(u_{m-3}) = 3)$ we have $n_{m-3}(4) = 1$, $n_{m-3}(2) = n_{m-3}(5) = \left\lfloor \dfrac{2n}{5} \right\rfloor$, $n_{m-3}(1) = \left\lceil \dfrac{n-1}{5} \right\rceil$.
- For vertex u_{m-2} $(f(u_{m-2}) = 4)$ we have $n_{m-2}(5) = 1$, $n_{m-2}(2) = n_{m-2}(3) = \left\lfloor \dfrac{2n}{5} \right\rfloor$, $n_{m-2}(1) = \left\lceil \dfrac{n-1}{5} \right\rceil$.

- For vertex u_{m-1} $(f(u_{m-1}) = 5)$ we have $n_{m-1}(3) = 1$, $n_{m-1}(1) = n_{m-1}(2) = \left\lfloor \frac{2n}{5} \right\rfloor$, $n_{m-1}(4) = \left\lceil \frac{n-1}{5} \right\rceil$.
- For vertex u_m $(f(u_m) = 1)$ we have $n_m(2) = 1$, $n_m(3) = n_m(4) = \left\lfloor \frac{2n}{5} \right\rfloor$, $n_m(5) = \left\lceil \frac{n-1}{5} \right\rceil$.

Each of the colors 1, 4 and 5 are used $2 + 2\lfloor 2n/5 \rfloor + 2\lceil (n-1)/5 \rceil$ times and colors 2 and 3 are used, each one with, $3 + 3\lfloor 2n/5 \rfloor$ times. For $n \bmod 5 \equiv 1$ or $n \bmod 5 \equiv 4$, the difference does not exceed one.

(c) $n \bmod 5 \equiv 2$

- For vertex u_{m-5} $(f(u_{m-5}) = 1)$ we have $n_{m-5}(2) = 1$, $n_{m-5}(3) = \frac{2n+1}{5}$, $n_{m-5}(4) = \frac{2n-4}{5}$, $n_{m-5}(5) = \frac{n-2}{5}$.
- For vertex u_{m-4} $(f(u_{m-4}) = 2)$ we have $n_{m-4}(3) = 1$, $n_{m-4}(1) = \frac{2n-4}{5}$, $n_{m-4}(5) = \frac{2n+1}{5}$, $n_{m-4}(4) = \frac{n-2}{5}$.
- For vertex u_{m-3} $(f(u_{m-3}) = 3)$ we have $n_{m-3}(4) = 1$, $n_{m-3}(2) = \frac{2n+1}{5}$, $n_{m-3}(5) = \frac{2n-4}{5}$, $n_{m-3}(1) = \frac{n-2}{5}$.
- For vertex u_{m-2} $(f(u_{m-2}) = 4)$ we have $n_{m-2}(5) = 1$, $n_{m-2}(2) = \frac{2n+1}{5}$, $n_{m-2}(3) = \frac{2n-9}{5}$, $n_{m-2}(1) = \frac{n+3}{5}$.
- For vertex u_{m-1} $(f(u_{m-1}) = 5)$ we have $n_{m-1}(3) = 1$, $n_{m-1}(1) = \frac{2n+1}{5}$, $n_{m-1}(2) = \frac{2n-9}{5}$, $n_{m-1}(4) = \frac{n+3}{5}$.
- For vertex u_m $(f(u_m) = 1)$ we have $n_m(2) = 1$, $n_m(3) = \frac{2n+1}{5}$, $n_m(4) = \frac{2n-4}{5}$, $n_m(5) = \frac{n-2}{5}$.

Each of the colors 1, 2 and 3 are used $(6n+3)/5+1$ times and colors 4 and 5 are used, each one with, $(6n+3)/5$ times.

(d) $n \bmod 5 \equiv 3$

- For vertex u_{m-5} $(f(u_{m-5}) = 1)$ we have $n_{m-5}(2) = 1$, $n_{m-5}(3) = n_{m-5}(4) = \frac{2n-1}{5}$, $n_{m-5}(5) = \frac{n-3}{5}$.
- For vertex u_{m-4} $(f(u_{m-4}) = 2)$ we have $n_{m-4}(3) = 1$, $n_{m-4}(1) = n_{m-4}(5) = \frac{2n-1}{5}$, $n_{m-4}(4) = \frac{n-3}{5}$.

- For vertex u_{m-3} ($f(u_{m-3}) = 3$) we have $n_{m-3}(4) = 1$, $n_{m-3}(2) = n_{m-3}(5) = \frac{2n-1}{5}$, $n_{m-3}(1) = \frac{n-3}{5}$.
- For vertex u_{m-2} ($f(u_{m-2}) = 4$) we have $n_{m-2}(5) = 1$, $n_{m-2}(2) = \frac{2n-1}{5}$, $n_{m-2}(3) = \frac{2n-6}{5}$, $n_{m-2}(1) = \frac{n+2}{5}$.
- For vertex u_{m-1} ($f(u_{m-1}) = 5$) we have $n_{m-1}(3) = 1$, $n_{m-1}(1) = \frac{2n-1}{5}$, $n_{m-1}(2) = \frac{2n-6}{5}$, $n_{m-1}(4) = \frac{n+2}{5}$.
- For vertex u_m ($f(u_m) = 1$) we have $n_m(2) = 1$, $n_m(3) = n_m(4) = \frac{2n-1}{5}$, $n_m(5) = \frac{n-3}{5}$.

Each of the colors 1, 2, 3 and 4 are used $(6n+2)/5+1$ times and color 5 is used $(6n+2)/5$ times.

In all the above cases the difference between the cardinalities of the color classes does not exceed one, so our coloring is equitable.

3. $m \bmod 5 \equiv 2$

We color first $m-2$ copies of W_n as we color the corresponding vertices in Case (1). We color last two copies (for u_{m-1} and u_m) in the following way. We consider five cases dependently on n.

(a) $n \bmod 5 \equiv 0$

- If $f(u_i) = 1$, $n_i(3) = 1$, $n_i(2) = \frac{2n-5}{5}$, $n_i(4) = \frac{2n-5}{5}$, $n_i(5) = \frac{n+5}{5}$.
- If $f(u_i) = 2$, $n_i(4) = 1$, $n_i(1) = n_i(3) = \frac{2n}{5}$, $n_i(5) = \frac{n-5}{5}$.

(b) $n \bmod 5 \equiv 1$ or $n \bmod 5 \equiv 4$

- If $f(u_i) = 1$, $n_i(3) = 1$, $n_i(2) = n_i(4) = \left\lfloor \frac{2n}{5} \right\rfloor$, $n_i(5) = \left\lceil \frac{n-1}{5} \right\rceil$.
- If $f(u_i) = 2$, $n_i(4) = 1$, $n_i(1) = n_i(3) = \left\lfloor \frac{2n}{5} \right\rfloor$, $n_i(5) = \left\lceil \frac{n-1}{5} \right\rceil$.

(c) $n \bmod 5 \equiv 2$

- If $f(u_i) = 1$, $n_i(3) = 1$, $n_i(2) = n_i(4) = \frac{2n-4}{5}$, $n_i(5) = \frac{n+3}{5}$.
- If $f(u_i) = 2$, $n_i(4) = 1$, $n_i(1) = n_i(3) = \frac{2n-4}{5}$, $n_i(5) = \frac{n+3}{5}$.

(d) $n \bmod 5 \equiv 3$

- If $f(u_i) = 1$, $n_i(3) = 1$, $n_i(2) = \frac{2n-6}{5}$, $n_i(4) = \frac{2n-1}{5}$, $n_i(5) = \frac{n+2}{5}$.

- If $f(u_i) = 2, n_i(4) = 1, n_i(1) = n_i(3) = \dfrac{2n-1}{5}, n_i(5) = \dfrac{n-3}{5}$.

In all the above cases the difference between the cardinalities of the color classes does not exceed one, so our coloring is equitable.

4. $m \bmod 5 = 3$
We color first $m-8$ copies of W_n as we have colored the corresponding vertices in Case (1). For each vertex u_i $(m-7 \le i \le m)$ we extend the coloring due to following conditions, dependently on n.
   (a) $n \bmod 5 \equiv 0$ or $n \bmod 5 \equiv 3$
      - For vertex u_{m-7} ($f(u_{m-7}) = 1$) we have $n_{m-7}(2) = 1$, $n_{m-7}(3) = \left\lfloor \dfrac{2n}{5} \right\rfloor$, $n_{m-7}(4) = \left\lfloor \dfrac{2n}{5} \right\rfloor - 1$, $n_{m-7}(5) = \left\lceil \dfrac{n}{5} \right\rceil$.
      - For vertex u_{m-6} ($f(u_{m-6}) = 2$) we have $n_{m-6}(1) = 1$, $n_{m-6}(3) = \left\lfloor \dfrac{2n}{5} \right\rfloor - 1$, $n_{m-6}(4) = \left\lfloor \dfrac{2n}{5} \right\rfloor$, $n_{m-6}(5) = \left\lceil \dfrac{n}{5} \right\rceil$.
      - For vertex u_{m-5} ($f(u_{m-5}) = 3$) we have $n_{m-5}(4) = 1$, $n_{m-5}(1) = \left\lfloor \dfrac{2n}{5} \right\rfloor$, $n_{m-5}(2) = \left\lfloor \dfrac{2n}{5} \right\rfloor - 1$, $n_{m-5}(5) = \left\lceil \dfrac{n}{5} \right\rceil$.
      - For vertex u_{m-4} ($f(u_{m-4}) = 4$) we have $n_{m-4}(3) = 1$, $n_{m-4}(1) = \left\lfloor \dfrac{2n}{5} \right\rfloor - 1$, $n_{m-4}(2) = \left\lfloor \dfrac{2n}{5} \right\rfloor$, $n_{m-4}(5) = \left\lceil \dfrac{n}{5} \right\rceil$.
      - For vertex u_{m-3} ($f(u_{m-3}) = 5$ we have $n_{m-3}(1) = 1$, $n_{m-3}(2) = \left\lfloor \dfrac{2n}{5} \right\rfloor$, $n_{m-3}(3) = \left\lfloor \dfrac{2n}{5} \right\rfloor - 1$, $n_{m-3}(4) = \left\lceil \dfrac{n}{5} \right\rceil$.
      - For vertex u_{m-2} ($f(u_{m-2}) = 1$) we have $n_{m-2}(2) = 1$, $n_{m-2}(3) = \left\lfloor \dfrac{2n}{5} \right\rfloor$, $n_{m-2}(5) = \left\lfloor \dfrac{2n}{5} \right\rfloor - 1$, $n_{m-2}(4) = \left\lceil \dfrac{n}{5} \right\rceil$.
      - For vertex u_{m-1} ($f(u_{m-1}) = 2$) we have $n_{m-1}(3) = 1$, $n_{m-1}(1) = \left\lfloor \dfrac{2n}{5} \right\rfloor - 1$, $n_{m-1}(5) = \left\lfloor \dfrac{2n}{5} \right\rfloor$, $n_{m-1}(4) = \left\lceil \dfrac{n}{5} \right\rceil$.
      - For vertex u_m ($f(u_m) = 3$) we have $n_m(5) = 1$, $n_m(1) = \left\lfloor \dfrac{2n}{5} \right\rfloor$, $n_m(2) = \left\lfloor \dfrac{2n}{5} \right\rfloor - 1$, $n_m(4) = \left\lceil \dfrac{n}{5} \right\rceil$.

Each of the colors 1, 2 and 3 are used $2 + 4\lfloor 2n/5 \rfloor$ times and colors 4 and 5 are used, each one with, $2\lfloor 2n/5 \rfloor + 4\lceil n/5 \rceil + 1$ times. For $n \bmod 5 \equiv 0$ or $n \bmod 5 \equiv 3$, the difference does not exceed one.

(b) $n \bmod 5 \equiv 1$

- For vertex u_{m-7} $(f(u_{m-7}) = 1)$ we have $n_{m-7}(2) = 1$, $n_{m-7}(3) = n_{m-7}(4) = \dfrac{2n-2}{5}$, $n_{m-7}(5) = \dfrac{n-1}{5}$.
- For vertex u_{m-6} $(f(u_{m-6}) = 2)$ we have $n_{m-6}(1) = 1$, $n_{m-6}(3) = \dfrac{2n-2}{5}$, $n_{m-6}(4) = \dfrac{2n-7}{5}$, $n_{m-6}(5) = \dfrac{n+4}{5}$.
- For vertex u_{m-5} $(f(u_{m-5}) = 3)$ we have $n_{m-5}(4) = 1$, $n_{m-5}(1) = n_{m-5}(2) = \dfrac{2n-2}{5}$, $n_{m-5}(5) = \dfrac{n-1}{5}$.
- For vertex u_{m-4} $(f(u_{m-4}) = 4)$ we have $n_{m-4}(3) = 1$, $n_{m-4}(1) = \dfrac{2n-2}{5}$, $n_{m-4}(2) = \dfrac{2n-7}{5}$, $n_{m-4}(5) = \dfrac{n+4}{5}$.
- For vertex u_{m-3} $(f(u_{m-3}) = 5)$ we have $n_{m-3}(1) = 1$, $n_{m-3}(2) = n_{m-3}(3) = \dfrac{2n-2}{5}$, $n_{m-3}(4) = \dfrac{n-1}{5}$.
- For vertex u_{m-2} $(f(u_{m-2}) = 1)$ we have $n_{m-2}(2) = 1$, $n_{m-2}(3) = \dfrac{2n-2}{5}$, $n_{m-2}(5) = \dfrac{2n-7}{5}$, $n_{m-2}(4) = \dfrac{n+4}{5}$.
- For vertex u_{m-1} $(f(u_{m-1}) = 2)$ we have $n_{m-1}(3) = 1$, $n_{m-1}(1) = n_{m-1}(5) = \dfrac{2n-2}{5}$, $n_{m-1}(4) = \dfrac{n-1}{5}$.
- For vertex u_m $(f(u_m) = 3)$ we have $n_m(5) = 1$, $n_m(1) = \dfrac{2n-7}{5}$, $n_m(2) = \dfrac{2n-2}{5}$, $n_m(4) = \dfrac{n+4}{5}$.

Each of the colors 1, 2, 4 and 5 are used $(8n+7)/5$ times and color 3 is used $(8n+7)/5 + 1$ times.

(c) $n \bmod 5 \equiv 2$

- For vertex u_{m-7} $(f(u_{m-7}) = 1)$ we have $n_{m-7}(2) = 1$, $n_{m-7}(3) = \dfrac{2n+1}{5}$, $n_{m-7}(4) = \dfrac{2n-4}{5}$, $n_{m-7}(5) = \dfrac{n-2}{5}$.
- For vertex u_{m-6} $(f(u_{m-6}) = 2)$ we have $n_{m-6}(1) = 1$, $n_{m-6}(3) = n_{m-6}(4) = \dfrac{2n-4}{5}$, $n_{m-6}(5) = \dfrac{n+3}{5}$.
- For vertex u_{m-5} $(f(u_{m-5}) = 3)$ we have $n_{m-5}(4) = 1$, $n_{m-5}(1) = \dfrac{2n+1}{5}$, $n_{m-5}(2) = \dfrac{2n-4}{5}$, $n_{m-5}(5) = \dfrac{n-2}{5}$.

- For vertex u_{m-4} ($f(u_{m-4}) = 4$) we have $n_{m-4}(3) = 1$, $n_{m-4}(1) = n_{m-4}(2) = \frac{2n-4}{5}$, $n_{m-4}(5) = \frac{n+3}{5}$.
- For vertex u_{m-3} ($f(u_{m-3}) = 5$) we have $n_{m-3}(1) = 1$, $n_{m-3}(2) = \frac{2n+1}{5}$, $n_{m-3}(3) = \frac{2n-4}{5}$, $n_{m-3}(4) = \frac{n-2}{5}$.
- For vertex u_{m-2} ($f(u_{m-2}) = 1$) we have $n_{m-2}(2) = 1$, $n_{m-2}(3) = n_{m-2}(5) = \frac{2n-4}{5}$, $n_{m-2}(4) = \frac{n+3}{5}$.
- For vertex u_{m-1} ($f(u_{m-1}) = 2$) we have $n_{m-1}(3) = 1$, $n_{m-1}(1) = \frac{2n-4}{5}$, $n_{m-1}(5) = \frac{2n+1}{5}$, $n_{m-1}(4) = \frac{n-2}{5}$.
- For vertex u_m ($f(u_m) = 3$) we have $n_m(5) = 1$, $n_m(1) = n_m(2) = \frac{2n-4}{5}$, $n_m(4) = \frac{n+3}{5}$.

Each of the colors 1, 2, 3 and 5 are used $(8n+4)/5+1$ times and color 4 is used $(8n+4)/5$ times.

(d) $n \bmod 5 \equiv 4$

- For vertex u_{m-7} ($f(u_{m-7}) = 1$) we have $n_{m-7}(2) = 1$, $n_{m-7}(3) = n_{m-7}(4) = \frac{2n-3}{5}$, $n_{m-7}(5) = \frac{n+1}{5}$.
- For vertex u_{m-6} ($f(u_{m-6}) = 2$) we have $n_{m-6}(1) = 1$, $n_{m-6}(3) = n_{m-6}(4) = \frac{2n-3}{5}$, $n_{m-6}(5) = \frac{n+1}{5}$.
- For vertex u_{m-5} ($f(u_{m-5}) = 3$) we have $n_{m-5}(4) = 1$, $n_{m-5}(1) = n_{m-5}(2) = \frac{2n-3}{5}$, $n_{m-5}(5) = \frac{n+1}{5}$.
- For vertex u_{m-4} ($f(u_{m-4}) = 4$) we have $n_{m-4}(3) = 1$, $n_{m-4}(1) = n_{m-4}(2) = \frac{2n-3}{5}$, $n_{m-4}(5) = \frac{n+1}{5}$.
- For vertex u_{m-3} ($f(u_{m-3}) = 5$) we have $n_{m-3}(1) = 1$, $n_{m-3}(2) = n_{m-3}(3) = \frac{2n-3}{5}$, $n_i(4) = \frac{n+1}{5}$.
- For vertex u_{m-2} ($f(u_{m-2}) = 1$) we have $n_{m-2}(2) = 1$, $n_{m-2}(3) = n_{m-2}(5) = \frac{2n-3}{5}$, $n_{m-2}(4) = \frac{n+1}{5}$.
- For vertex u_{m-1} ($f(u_{m-1}) = 2$) we have $n_{m-1}(3) = 1$, $n_{m-1}(1) = n_{m-1}(5) = \frac{2n-3}{5}$, $n_{m-1}(4) = \frac{n+1}{5}$.
- For vertex u_m ($f(u_m) = 3$) we have $n_m(5) = 1$, $n_m(1) = n_m(2) = \frac{2n-3}{5}$, $n_m(4) = \frac{n+1}{5}$.

Each of the colors are used $(8n+8)/5$ times.

In all the above cases the difference between the cardinalities of the color classes does not exceed one, so our coloring is equitable.

5. $m \bmod 5 \equiv 4$
   We color first $m-4$ copies of W_n as we have colored the corresponding vertices in Case (1). Then, we color last four copies in the following way. For each vertex u_i, $(m-3 \leq i \leq m)$, we color the corresponding copy of W_n due the following conditions, dependently on n.
   (a) $n \bmod 5 \equiv 0$
   If $f(u_i) = j$, $1 \leq j \leq 4$, then
   - $n_i((j+1) \bmod 4) = 1$,
   - $n_i((j+2) \bmod 4) = \dfrac{2n}{5}$,
   - $n_i((j+3) \bmod 4) = \dfrac{2n-5}{5}$,
   - $n_i(5) = \dfrac{n}{5}$.

   (b) $n \bmod 5 \equiv 1$
   For vertex u_{m-3} ($f(u_{m-3}) = 1$) we have $n_{m-3}(2) = 1, n_{m-3}(3) = \dfrac{2n-2}{5}$, $n_{m-3}(4) = \dfrac{2n-7}{5}, n_{m-3}(5) = \dfrac{n+4}{5}$.
   For vertices u_i, $m-2 \leq i \leq m$, if $f(u_i) = j$, $1 \leq j \leq 4$, then
   - $n_i((j+1) \bmod 4) = 1$,
   - $n_i((j+2) \bmod 4) = n_i((j+3) \bmod 4) = \dfrac{2n-2}{5}$,
   - $n_i(5) = \dfrac{n-1}{5}$.

   (c) $n \bmod 5 \equiv 2$
   For vertices u_i, $m-3 \leq i \leq m-2$, if $f(u_i) = j$, $1 \leq j \leq 2$, then
   - $n_i((j+1) \bmod 4) = 1$,
   - $n_i((j+2) \bmod 4) = n_i((j+3) \bmod 4) = \dfrac{2n-4}{5}$,
   - $n_i(5) = \dfrac{n+3}{5}$.

   For vertices u_i, $m-1 \leq i \leq m$, if $f(u_i) = j$, $3 \leq j \leq 4$, then
   - $n_i((j+1) \bmod 4) = 1$,
   - $n_i((j+2) \bmod 4) = \dfrac{2n+1}{5}$,
   - $n_i((j+3) \bmod 4) = \dfrac{2n-4}{5}$,
   - $n_i(5) = \dfrac{n-2}{5}$.

   (d) $n \bmod 5 \equiv 3$
   If $f(u_i) = j$, $1 \leq j \leq 4$, then

- $n_i((j+1) \bmod 4) = 1$,
- $n_i((j+2) \bmod 4) = \dfrac{2n-1}{5}$,
- $n_i((j+3) \bmod 4) = \dfrac{2n-6}{5}$,
- $n_i(5) = \dfrac{n+2}{5}$.

(e) $n \bmod 5 \equiv 4$
If $f(u_i) = j$, $1 \leq j \leq 4$, then
- $n_i((j+1) \bmod 4) = 1$,
- $n_i((j+2) \bmod 4) = n_i((j+3) \bmod 4) = \dfrac{2n-3}{5}$,
- $n_i(5) = \dfrac{n+1}{5}$.

In all the above cases the difference between the cardinalities of the color classes does not exceed one, so our coloring is equitable. Hence $\chi_=(G \circ W_n) \leq 5$. By the definition of corona graph for each vertex u_i of G, there exists a copy of W_n whose vertices are adjacent to the vertex u_i.

Case 1: If $m \mod 2 \equiv 1, n \geq 7$

In this case either both m and n are odd (or) m is odd and n is even.

(a) If m and n are odd.
Since $\chi(W_n) = 3$ for odd n, we need at least 4 colors for coloring each copy of W_n and the corresponding vertex of G. In this coloring, since m is odd there exists atleast one color which reappears in $\langle \{u_i : 1 \leq i \leq m\} \rangle$. Let the color $j\,(1 \leq j \leq 4)$ reappears at the vertex $u_i\,(5 \leq i \leq m)$. Then the center vertex of the copy W_n corresponding to the vertex u_i, receives a color $k\,(1 \leq k \leq 4)$, where $k \neq j$. Other vertices of W_n receive the colors other than j and k. (i.e) The number of possible colors to color these vertices is two. Hence it is clear that for the case of $n \geq 5$, it is not possible to color the vertices of the cycle C_{n-1} of W_n equitably with two colors. Therefore $\chi_=(G \circ W_n) \geq 5$. Hence $\chi_=(G \circ W_n) = 5$ for m and n are odd.

(b) If m is odd and n is even.
Since $\chi\,(W_n) = 4$ for even n, the graph $G \circ W_n$ requires at least 5 colors. Hence $\chi_=(G \circ W_n) = 5$ for m is odd and n is even.

Case 2: If m and n are even, $n \geq 4$

Since $\chi(W_n) = 4$ for even n, graph $G \circ W_n$ requires at least 5 colors. Therefore $\chi_=(G \circ W_n) \geq 5$.

Hence $\chi_=(G \circ W_n) = 5$ for even n.

$\square$

Next, we consider coronas, where the set of vertices of graph G includes exactly three elements.

Theorem 2.5. *Let G be an equitably 3-colorable graph with $m = 3$ vertices. Then*

1. $\chi_=(G \circ W_5) = 4$.
2. $\chi_=(G \circ W_n) = 5 \;\; n = 7, 9, 11, 13, 15, 17$.
3. $\chi_=(G \circ W_n) = 5 \;\; n \geq 19$*, if n is odd.*
4. $\chi_=(G \circ W_n) = 5 \;\; n = 4, 6, 8, 10$.
5. $\chi_=(G \circ W_n) = 6 \;\; n \geq 12$*, if n is even.*

Proof. Let $\{u_i : 1 \leq i \leq 3\}$ be the set of vertices of G.

1. We color $G \circ W_5$ as for the following procedure.
   - For vertex $u_1\,(f(u_1) = 1)$ we have $n_1\,(2) = 1$, $n_1\,(3) = n_1\,(4) = 2$.
   - For vertex $u_2\,(f(u_2) = 2)$ we have $n_2\,(3) = 1$, $n_2\,(4) = n_2\,(1) = 2$.
   - For vertex $u_3\,(f(u_3) = 3)$ we have $n_3\,(4) = 1$, $n_3\,(1) = n_3\,(2) = 2$.

   In the above cases the difference between the cardinalities of the color classes does not exceed one, so our coloring is equitable. Hence $\chi_=(G \circ W_5) \leq 4$. Since W_5 is 3-colorable, at each copy of W_5 of $G \circ W_5$, there exists one more color. Therefore $\chi_=(G \circ W_5) \geq 4$. hence $\chi_=(G \circ W_5) = 4$.
2. Assign the color i to the vertex $u_i\,(1 \leq i \leq 3)$, color 4 to the vertex u_{1n}, color 5 to the vertex u_{2n} and color 1 to the vertex u_{3n}. Since C_{n-1} is of even order, we require only two colors for proper coloring of C_{n-1}. We use three colors in each C_{n-1} of W_n in $G \circ W_n$. We use the colors 2,3,5 to the vertices of C_{n-1} of W_n at u_1. Similarly we use the colors 1,3,4 and 4,5,2 to the vertices of C_{n-1} of W_n at u_2 and u_3 respectively. The number of appearance of the colors are given in the following cases.
   (a) $n = 7, 17$
      - For vertex $u_1\,(f\,(u_1 = 1))$ we have $n_1\,(4) = 1$, $n_1\,(2) = \dfrac{2n+1}{5}$, $n_1\,(3) = \dfrac{2n-4}{5}$, $n_1\,(5) = \dfrac{n-2}{5}$.
      - For vertex $u_2\,(f\,(u_2 = 2))$ we have $n_2\,(5) = 1$, $n_2\,(1) = \dfrac{n-1}{2}$, $n_2\,(3) = \left\lceil \dfrac{n-1}{4} \right\rceil$, $n_2\,(4) = \left\lfloor \dfrac{n-1}{4} \right\rfloor$.
      - For vertex $u_3\,(f\,(u_3 = 3))$ we have $n_3\,(1) = 1$, $n_3\,(4) = \dfrac{2n-4}{5}$, $n_3\,(5) = \dfrac{2n+1}{5}$, $n_3\,(2) = \dfrac{n-2}{5}$.

   (b) $n = 9$
      - For vertex $u_1\,(f\,(u_1) = 1)$ we have $n_1\,(4) = 1$, $n_1\,(2) = n_1\,(3) = 3$, $n_1\,(5) = 2$.
      - For vertex $u_2\,(f\,(u_2) = 2)$ we have $n_2\,(5) = 1$, $n_2\,(1) = 4$, $n_2\,(3) = n_2\,(4) = 2$.

- For vertex $u_3\,(f\,(u_3) = 3)$ we have $n_3\,(1) = 1$, $n_3\,(4) = n_3\,(5) = 3$, $n_3\,(2) = 2$.

(c) $n = 11$

- For vertex $u_1\,(f\,(u_1) = 1)$ we have $n_1\,(4) = 1$, $n_1\,(2) = n_1\,(3) = 4$, $n_1\,(5) = 2$.
- For vertex $u_2\,(f\,(u_2) = 2)$ we have $n_2\,(5) = 1$, $n_2\,(1) = 5$, $n_2\,(3) = 3$, $n_2\,(4) = 2$.
- For vertex $u_3\,(f\,(u_3) = 3)$ we have $n_3\,(1) = 1$, $n_3\,(4) = n_3\,(5) = 4$, $n_3\,(2) = 2$.

(d) $n = 13$

- For vertex $u_1\,(f\,(u_1) = 1)$ we have $n_1\,(4) = 1$, $n_1\,(2) = n_1\,(3) = 5$, $n_1\,(5) = 2$.
- For vertex $u_2\,(f\,(u_2) = 2)$ we have $n_2\,(5) = 1$, $n_2\,(1) = 6$, $n_2\,(3) = 3$, $n_2\,(4) = 3$.
- For vertex $u_3\,(f\,(u_3) = 3)$ we have $n_3\,(1) = 1$, $n_3\,(4) = n_3\,(5) = 5$, $n_3\,(2) = 2$.

(e) $n = 15$

- For vertex $u_1\,(f\,(u_1) = 1)$ we have $n_1\,(4) = 1$, $n_1\,(2) = n_1\,(3) = 6$, $n_1\,(5) = 2$.
- For vertex $u_2\,(f\,(u_2) = 2)$ we have $n_2\,(5) = 1$, $n_2\,(1) = 7$, $n_2\,(3) = 3$, $n_2\,(4) = 4$.
- For vertex $u_3\,(f\,(u_3) = 3)$ we have $n_3\,(1) = 1$, $n_3\,(4) = 4$, $n_3\,(5) = 7$, $n_3\,(2) = 2$.

In the above cases the difference between the cardinalities of the color classes does not exceed one, so our coloring is equitable. Hence $\chi_=(G \circ W_n) \leq 5$.
Since G is 3-colorable, let i be the color assigned to the vertex $u_i\,(1 \leq i \leq 3)$ of $G \circ W_n$. Let $j\,(1 \leq j \leq 4)$, $(i \neq j)$ be the color assigned to the center vertices of each copy W_n of $G \circ W_n$. The other vertices of these copies receive the colors other than i and j. (i.e) The number of possible colors to color these vertices is two. Hence it is clear that for the case of $n = 7, 9, 11, 13, 15, 17$, it is not possible to color the vertices of the cycle C_{n-1} of W_n equitably with two colors. Therefore $\chi_=(G \circ W_n) \geq 5$. Hence $\chi_=(G \circ W_n) = 5$ for $n = 7, 9, 11, 13, 15, 17$.

3. Suppose that $G \circ W_n$ is $4-$equitably colorable. Since G is $3-$colorable, let it be colored by the color 1,2 and 3. Let u_i receives the color $i\,(1 \leq i \leq 3)$. Then u_{1n}, u_{2n} and u_{3n} should receive any two of the three color 1,2,3 and the color 4.
Let u_{1n} receive 4, u_{2n} receive 1 and u_{2n} receive 2. Then $u_{1i}\,(1 \leq i \leq n-1)$ receives the color 2, $\dfrac{n-1}{2}$ times and 3, $\dfrac{n-1}{2}$ times. $u_{2i}\,(1 \leq i \leq n-1)$ receives the color 3, $\dfrac{n-1}{2}$ times, the color 4, $\dfrac{n-1}{2}$ times. Similarly u_{3i} receives the color 1, $\dfrac{n-1}{2}$ times and the color 1, $\dfrac{n-1}{2}$ times.
Number of appearance of each colors 1 and 2 are, $\dfrac{n+3}{2}$ times respectively and number of appearance of each colors 3 and 4 are, n times respectively.
As the above mentioned partition does not imply the equitable partition, it is concluded that $G \circ W_n$ should not be equitable 4-colorable.
Hence $\chi_=\,(G \circ W_n) \geq 5$
Suppose that $G \circ W_n$ is $5-$equitable colorable. Let G be colored by the colors 1, 2 and 3. Let u_i receives the color $i\;(1 \leq i \leq 3)$. Since $G \circ W_n$ is $5-$equitable colorable, any two of the vertices u_{1n}, u_{2n} and u_{3n} receives the color 4 and 5 (Say u_{1n}, u_{2n}) and remaining vertex u_{3n} should receive the color 1.

For the case of $n \geq 19$, if we use the above coloring with 5 colors, then the maximum of appearance of color 1, $\frac{n-1}{2} + 2 = \frac{n+3}{2}$ times.
Remaining number of vertices to be colored are, $3n + 3 - \frac{n+3}{2} = \frac{5n+3}{2}$.
Number of vertices which receive each colors of $2, 3, 4$ and 5 are $\frac{\frac{5n+3}{2}}{4} = \frac{5n+3}{8}$.
For $n \geq 19$, $\left\lceil \frac{5n+3}{2} \right\rceil - \left\lceil \frac{n+3}{2} \right\rceil \geq 2$.
(i.e) it may not be possible to equitably color $G \circ W_n$ with 5 colors.
$\chi_= (G \circ W_n) \geq 6$.

- For vertex $u_1\,(f\,(u_1) = 1)$ we have $n_1\,(4) = 1$, $n_1\,(2) = n_1\,(3) = \frac{n-1}{2}$.
- For vertex $u_2\,(f\,(u_2) = 2)$ we have $n_2\,(5) = 1$, $n_2\,(1) = n_2\,(6) = \frac{n-1}{2}$.
- For vertex $u_3\,(f\,(u_3) = 3)$ we have $n_3\,(6) = 1$, $n_3\,(5) = n_3\,(4) = \frac{n-1}{2}$.

In the above cases the difference between the cardinalities of color classes does not exceed one, so our coloring is equitable. Hence $\chi_= (G \circ W_n) = 6$, $n \geq 19$, if n is odd.

4. Since n is even W_n has odd cycle C_{n-1}. Minimum number of colors assigned to color any cycle is 3. Hence $u_{in}\,(1 \leq i \leq n)$ should have a fourth color and hence $u_i\,(1 \leq i \leq n)$ must receive a fifth color. Hence $\chi_= (G \circ W_n) \geq 5$.
Now we partition the vertex set $V\,(G \circ W_n)$ as follows,

$$\begin{aligned}
V_1 &= \{u_1, u_{21}, u_{23}, u_{25}, u_{28}, u_{3n}\} \\
V_2 &= \{u_2, u_{11}, u_{14}, u_{18}, u_{33}, u_{36}, u_{39}\} \\
V_3 &= \{u_3, u_{12}, u_{15}, u_{17}, u_{24}, u_{27}\} \\
V_4 &= \{u_{1n}, u_{22}, u_{26}, u_{29}, u_{32}, u_{35}, u_{37}\} \\
V_5 &= \{u_{2n}, u_{13}, u_{16}, u_{19}, u_{31}, u_{34}, u_{38}\}
\end{aligned}$$

Clearly V_1, V_2, V_3, V_4 and V_5 are independent set of $G \circ W_n$. Hence $||V_i| - |V_j|| \leq 1$ for every $i \neq j$. Hence $\chi_= (G \circ W_n) = 5$, $4 \leq n \leq 10$, if n is even.

5. Let $n_i(k)$ be the number of appearance of the color k in the copy of W_n corresponding to the vertex u_i of G in $G \circ W_n$.
Let $f(u_i) = j$ be the color assigned to each vertices $u_i (1 \leq i \leq m)$ of G. Since G is 6-colorable j takes the values in the range $1 \leq j \leq 6$.

   - For vertex $u_1\,(f\,(u_1) = 1)$ we have $n_1\,(2) = n_1\,(5) = 1$, $n_1\,(3) = n_1\,(4) = \frac{n-2}{2}$.
   - For vertex $u_2\,(f\,(u_2) = 2)$ we have $n_2\,(3) = n_2\,(1) = 1$, $n_2\,(5) = n_2\,(6) = \frac{n-2}{2}$.
   - For vertex $u_3\,(f\,(u_3) = 3)$ we have $n_3\,(6) = n_3\,(4) = 1$, $n_3\,(1) = n_3\,(2) = \frac{n-2}{2}$.

In the above cases the difference between the cardinalities of the color classes does not exceed one, so our coloring is equitable. Hence $\chi_=(G \circ W_n) \leq 6$
Since n is even, we require at least 3 colors to color each C_{n-1} of W_n, one color for the centre vertex of W_n and one color corresponding to the vertex of G. Hence we may assume that $\chi_=(G \circ W_n) = 5$. It is clear that one of these five colors appears twice in $\langle\{u_i : 1 \leq i \leq 3\} \bigcup \{u_{in} : 1 \leq i \leq 3\}\rangle$, let it be color $j\,(1 \leq j \leq 5)$. This color j can be assigned only $\dfrac{n-2}{2}$ times in any of the C_{n-1} copy of W_{n-1}. This violate the equitable conclusion.

Therefore $\chi_=(G \circ W_n) \geq 6$. Hence $\chi_=(G \circ W_n) = 6$. □

3. Conclusion

We notice that the results can be extended into further products of graphs.

Corollary 3.1. *Let G be an equitably 4-colorable graph on, $m \geq 2$, vertices, let m is even, n is odd, and $n \geq 4$, and $l \geq 1$. Then*

$$\chi^l_=(G \circ W_n) = 4.$$

Proof. We use the principle of mathematical induction due to number l.

1. l=1
   The truth follows immediately from Theorem 2.2.
2. Induction hypothesis for l. It means that $\chi_=(G \circ^l W_n) = 4$ for n odd and $m = |V(G)|$ even.
3. We must show that $\chi_=(G \circ^{l+1} W_n) = 4$ for graphs under consideration.

Let us notice that graph from induction hypothesis $G \circ^l W_n$ is an equitably 4-colorable graph, it means a graph fulfilling the assumption of Theorem 2.2. Its number of vertices, equals to $m(n+1)^l$ is an even number for m even. So, $\chi_=(G \circ^{l+1} W_n) = 4$. □

Corollary 3.2. *Let G be an equitably 5-colorable graph on m vertices and let $m \geq 2$, $n \geq 4$, $l \geq 1$. Then*

$$\chi_=(G \circ^l W_n) = \begin{cases} = 5 & \text{for } n \text{ even}, \\ \leq 5 & \text{for } m \text{ and } n \text{ odd}. \end{cases}$$

Proof. Follows immediately from Theorem 2.4. □

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